The Total Differential dz

Let $z = f(x, y)$ be a function of two variables.

At the beginning of a particular process, the values of x and y are set at $x = x_0$ and $y = y_0$.

Thus, the INITIAL INPUT is (x_0, y_0) and the INITIAL OUTPUT is $z_0 = f(x_0, y_0)$.

During the process, the values of x and y change by the amounts Δx and Δy, resp., so that *at the end of the process:*

the FINAL INPUT is (x_1, y_1) and the FINAL OUTPUT is $z_1 = f(x_1, y_1)$.

(Here, $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$)

The resulting *change* in the OUTPUT $z = f(x, y)$ value is notated Δf or Δz and is calculated as follows:

$\Delta z = \Delta f =$ (FINAL OUTPUT) – (INITIAL OUTPUT), that is,

$$\Delta z = \Delta f = f(x_1, y_1) - f(x_0, y_0) = z_1 - z_0 .$$

For Example, consider the following process:

PROCESS: A rectangular mat starts out with dimensions 20 inches $\times$ 30 inches .

The mat is stretched uniformly so that the new dimensions are 20.2 inches $\times$ 30.5 inches .

The area $z = A = f(x, y)$ is a function of the height x (inches) and width y (inches) of the rectangle, according to the area function: Area $A = z = \mathbf{f(x, y) = xy}$ square inches .

The INITIAL INPUT is $(x_0, y_0) = (20, 30)$ and

the INITIAL OUTPUT is $z_0 = f(x_0, y_0) = f(20, 30) = (20)(30) = 600$ square inches .

The FINAL INPUT is $(x_1, y_1) = (20.2, 30.5)$ and

the FINAL OUTPUT: $z_1 = f(x_1, y_1) = f(20.2, 30.5) = (20.2)(30.5) = 616.1$ sq. in.

Here, $\Delta x = 0.2$ inches , $\Delta y = 0.5$ inches , and $\Delta z = 16.1$ sq. in.

The Total Differential **dz** gives a formula for *approximating* Δz in terms of x_0, y_0, Δx, and Δy.

Definition: Given the function $z = f(x,y)$,

the **Total Differential dz** at the point (x,y) is the expression

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy \quad \text{or, equivalently,} \quad dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy,$$

where, in any particular application, $x = x_0$, $y = y_0$, $dx = \Delta x$ *and* $dy = \Delta y$.

For example, let the function f be the function $z = f(x,y) = x^2 y^3 - y^4$,

which has partial derivatives $\frac{\partial z}{\partial x} = 2xy^3$ and $\frac{\partial z}{\partial y} = 3x^2y^2 - 4y^3$.

In this case, the Total Differential **dz** is the following: $dz = (2xy^3)\,dx + (3x^2y^2 - 4y^3)\,dy$

The Total Differential Principle says that, as a result of some process which changes the values of x and y from the initial values (x_0, y_0) to the final values (x_1, y_1), the change in the f value, $\Delta z = \Delta f$, can be approximated by the value of the Total Differential **dz** when it is calculated using the initial inputs (x_0, y_0) and using $dx = \Delta x$ *and* $dy = \Delta y$, that is,

$$\Delta z = \Delta f \approx dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

where $x = x_0$, $y = y_0$, $dx = \Delta x$ *and* $dy = \Delta y$.

Example: Consider the function $z = f(x,y) = 5x^3 y + xy^2$.

$$f_x(x,y) = \frac{\partial z}{\partial x} = 15x^2y + y^2 \quad \text{and} \quad f_y(x,y) = \frac{\partial z}{\partial y} = 5x^3 + 2xy.$$

The Total Differential **dz** is **dz** $= (15x^2y + y^2)\,dx + (5x^3 + 2xy)\,dy$.

Let the INITIAL INPUTS be $(x_0, y_0) = (1, 2)$ and the FINAL INPUTS be $(x_1, y_1) = (1.3, 1.9)$.

Here, $\Delta x = x_1 - x_0 = 0.3$ and $\Delta y = y_1 - y_0 = -0.1$.

Here, the Total Differential **dz** is **dz** $= (34)(0.3) + (9)(-0.1) = 10.2 - 0.9 = 9.3$.

According to the Total Differential Principle, **dz** is an approximation to the change in z, Δz.

Here the actual change in z is $\Delta z = f(1.3, 1.9) - f(1, 2) = 11.5645$.

Thus, **dz** $= 9.3$ approximates $\Delta z = 11.5645$, which is a change in z from changing inputs from the initial values $(x_0, y_0) = (1, 2)$.

Recall the example from page 1:

PROBLEM: A rectangular mat starts out with dimensions 20 inches $\times$ 30 inches .

The area of the mat A = f(x , y) is a function of the height x (inches) and width y (inches) of the rectangle, according to the area function: Area A = **f(x , y)** = **x y** square inches .

Task: Using the Total Differential dA ,

approximate ΔA = the increase in the area A which results when the mat is stretched uniformly so that the new dimensions are 20.2 inches $\times$ 30.5 inches .

Solution: The initial value of (x , y) is (x_0, y_0) = (20 , 30) .

The final value of (x , y) is (x_1, y_1) = (20.2 , 30.5) .

The change in the values of the variables are $\Delta x = 0.2$ and $\Delta y = 0.5$.

The partial derivatives of A are $\dfrac{\partial A}{\partial x} = y$ and $\dfrac{\partial A}{\partial y} = x$, so the Total Differential dA is:

$$dA = \frac{\partial A}{\partial x}\, dx + \frac{\partial A}{\partial y}\, dy = (\, y\,)\, dx + (\, x\,)\, dy \;.$$

According to the Total Differential Principle,

The increase in area $\Delta A \approx dA = f_x(x, y)\, dx + f_y(x, y)\, dy$

That is, $\Delta A \approx dA = (y)\,dx + (x)\,dy$.

where $x = x_0$, $y = y_0$, $dx = \Delta x$ *and* $dy = \Delta y$.

Here, $x_0 = 20$, $y_0 = 30$, $\Delta x = 0.2$ and $\Delta y = 0.5$.

Thus, $\Delta A \approx dA = (30)(0.2) + (20)(0.5) = 16$ square inches.

The Total Differential dz approximates the increase in area A as being about 16 square inches.

The actual increase in area A is $\Delta z = f(20.2\,,\ 30.5) - f(20,\ 30) = 16.1$ square inches.